# The first three rungs of the cosmological distance ladder

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It is straightforward to determine the size of the Earth and the distance to the Moon without using a telescope. The methods have been known since the third century BCE. However, few astronomers have done this measurement from data they have taken. We use a gnomon to determine the latitude and longitude of South Bend, Indiana, and College Station, Texas, and determine the value of the radius of the Earth to be $R_{\mathrm{earth}} = 6290$ km, only 1.4% smaller than the known value. We use the method of Aristarchus and the size of the Earth's shadow during the lunar eclipse of June 15, 2011 to estimate the distance to the Moon to be $62.3R_{\mathrm{earth}}$, 3.3% greater than the known mean value. We use measurements of the angular motion of the Moon against the background stars over the course of two nights, using a simple cross staff device, to estimate the Moon's distance at perigee and apogee. We use simultaneous observations of asteroid 1996 HW1 obtained with small telescopes in Socorro, New Mexico, and Ojai, California, to obtain a value of the Astronomical Unit of $(1.59 \pm 0.19) \times 10^8$ km, about 6% too large. The data and methods presented here can easily become part of an introductory astronomy laboratory class. 
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## I. INTRODUCTION

If we had to derive everything we use from scratch, we would not have the time to make much progress. But it is worthwhile to derive some fundamental parameters that we use in our area of expertise. In 1987, Philip Morrison hosted a television series called "Ring of truth: An inquiry into how we know what we know." One purpose of the show was to demonstrate simple measurements. In the episode "Mapping" he measured the circumference of the Earth with the "van of Eratosthenes."[1] He and coworkers went to the north edge of Kansas and measured the elevation angle of the bright southern star Antares as it transited the meridian. Then, they drove 370 miles to the south edge of Kansas and measured Antares again. They found that it was 5° higher at the second location. The distance corresponds to $1/72$ of a circle, so the implied circumference of the Earth is 26,600 miles, which is a little more than 6% greater than the known value of 24,901 miles.

The cosmological distance ladder is a sequence of steps used by astronomers to derive distances within the solar system, throughout the galaxy, and beyond to the farthest galaxies detectable.[2] It hinges on simple geometry and the principles of surveying. For example, using the positions of two observers separated by some baseline on the Earth, we can determine the distance to the Moon, a nearby planet, or an asteroid.

In this paper, we determine the sizes of the first three rungs of the cosmological distance ladder, the radius of the Earth, the distance to the Moon, and the distance to the Sun. The first two distances can be obtained without a telescope. The third was attempted by the various pre-telescopic astronomers such as Tycho Brahe in 1582 (Ref. 3) but was not accomplished until 1672 by Gian Domenico Cassini and John Flamsteed,[4] who made observations of Mars when it was prominent in the middle of the night and therefore about as close to the Earth as it gets. (Mars is roughly 0.6 AU distant at such a time. When Mars is on the other side of the Sun it is 2.6 AU distant.) Measuring such distances requires telescopic measurements or distances to nearby planets or asteroids determined with radar.

Once the scale of the solar system is known, we can determine distances to nearby stars by trigonometric stellar parallaxes if our positional measurements are good to better than 0.1 arc sec. We then determine the distance to the Hyades star cluster[5] and relate other star clusters to the Hyades distance. With the discovery of certain standard candles and standardizable candles in star clusters (for example, RR Lyrae stars and Cepheids), we can calibrate the mean intrinsic brightness of these pulsating stars. They are useful for distance determinations throughout our galaxy. With the Hubble Space Telescope we can determine distances to other galaxies using the Cepheid period-luminosity relation as far

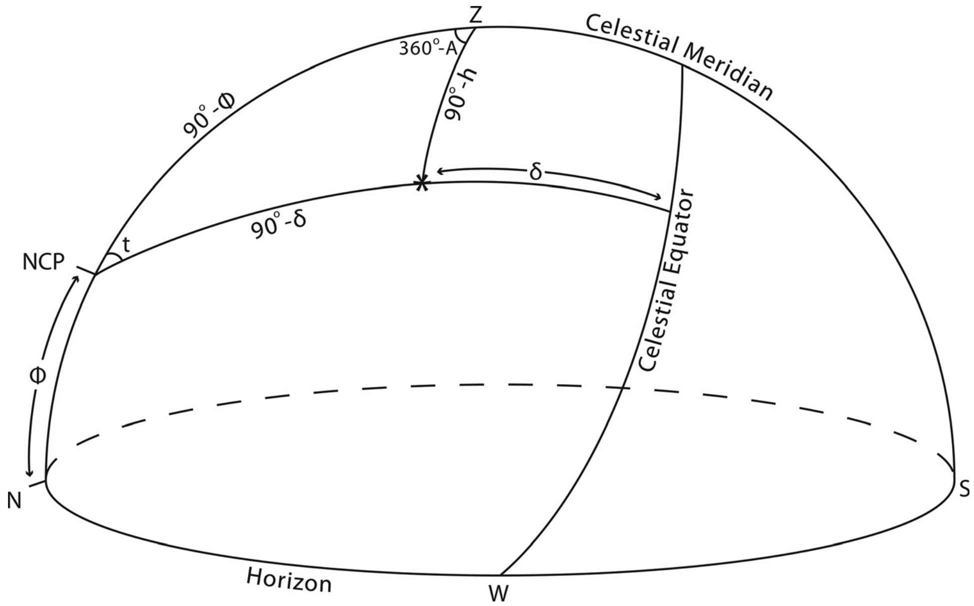


Fig. 1. The astronomical triangle. The North Celestial Pole is labeled NCP. The zenith is at Z. The latitude of the site is $\phi$. An object in the western sky is marked by an asterisk. The hour angle of the object is $t$, which is positive in the western sky, negative in the eastern sky. The declination of the object is $\delta$. Its elevation angle above the horizon is $h$, so the zenith angle is $90^\circ - h$. The azimuth A of an object is measured clockwise around the horizon, with north $= 0^\circ$.

as $25 \times 10^6$ parsecs.[6] (1 parsec equals $3.086 \times 10^{13}$ km, or 206265 AU.) If we observe Type Ia supernovae in some galaxies whose distances are known via Cepheids, we can calibrate the intrinsic brightness of these supernovae. Because Type Ia supernovae are typically $4 \times 10^9$ times brighter than the Sun at maximum brightness, they can be used to determine distances halfway across the observable universe with a 4-m class telescope.

In this paper we use our own data only, if possible. We make minimal use of the *Astronomical Almanac* (Ref. 7) and minimal use of telescopes. Many of the results here can be obtained by students using very unsophisticated and inexpensive equipment.

## II. DETERMINING ONE'S POSITION ON THE EARTH

The astronomical triangle is shown in Fig. 1. The northern sky turns around the North Celestial Pole near the direction of the star Polaris. The North Celestial Pole is $\phi$ degrees above the horizon; $\phi$ is the latitude. The azimuth $A$ is measured clockwise around the horizon and is equal to $0^\circ$ at the north point on the horizon, and is $90^\circ$ at the east point on the horizon. The hour angle $t$ divided by $15^\circ$/h is the number of hours that an object is west of the celestial meridian; $t$ is negative for objects in the eastern sky.

The declination $\delta$ of a celestial object is the number of degrees the object is north or south of the celestial equator. If we know the declination of the Sun and we determine how high it is above the horizon when it transits the *celestial meridian*,[8] we can determine our latitude. At local apparent noontime, the elevation angle of the Sun is

$$h_{\max} = 90^\circ - \phi + \delta. \tag{1}$$

We are concerned here with the Sun and Moon as observed at mid-northern latitudes. These objects transit the celestial meridian between the zenith and the south point on the horizon.

To determine the elevation angle of the Sun we can set up a vertical pointed stick or *gnomon*. Even better is a vertical stick with a small sphere at the top, like that shown in Fig. 2. It is easier to measure the center of the elliptical shadow of the sphere on the ground than it is to measure the end of the darker part of the shadow of a vertical pointed stick.[9]

The Sun's declination ranges from $-\epsilon$ to $+\epsilon$ over the course of the year. $\epsilon$ is the *obliquity of the ecliptic*, the tilt of the Earth's axis of rotation to the plane of its orbit. (The ecliptic is the apparent path of the Sun through the

Fig. 2. A gnomon consists of a wooden base with a hole drilled through it, and a vertical stick which fits tightly. It can be a pointed stick. We have fashioned a small sphere at the top. It is easier to measure the center of the elliptical shadow of the sphere than the end of the darker part of the shadow of a pointed stick.

constellations of the zodiac due to the Earth's orbit around the Sun.) On the first day of summer, we have

$$h_{\max}(\text{June } 21) = 90^\circ - \phi + \epsilon. \tag{2}$$

On December 21, we have

$$h_{\max}(\text{December } 21) = 90^\circ - \phi - \epsilon. \tag{3}$$

The sum of Eqs. (2) and (3) gives

$$\phi = 90^\circ - [h_{\max}(\text{June } 21) + h_{\max}(\text{June } 21)]/2. \tag{4}$$

Equation (2) minus Eq. (3) gives

$$\epsilon = [h_{\max}(\text{June } 21) - h_{\max}(\text{June } 21)]/2. \tag{5}$$

In other words, if we determine the maximum elevation angle of the Sun on June 21 and December 21, the average gives $90^\circ - \phi$, and half the difference gives $\epsilon$.

In Table I, we give the raw data from two solstice experiments done at College Station, Texas. In Fig. 3, we show the X-Y positions of the end of the gnomon's shadow for three key times of the year. Note that when the declination of the Sun is negative, the positions of the end of the shadow on the ground trace out an upward pointing curve. On the first day of spring (20 or 21 March) or the first day of autumn (about 23 September) the points delineate a straight line. When the declination of the Sun is positive, the points trace out a downward curving locus. A graph of this type allows us to determine which points, if any, have been mismeasured. If a student invents data for such an experiment, such a graph allows us to detect this fraud easily.

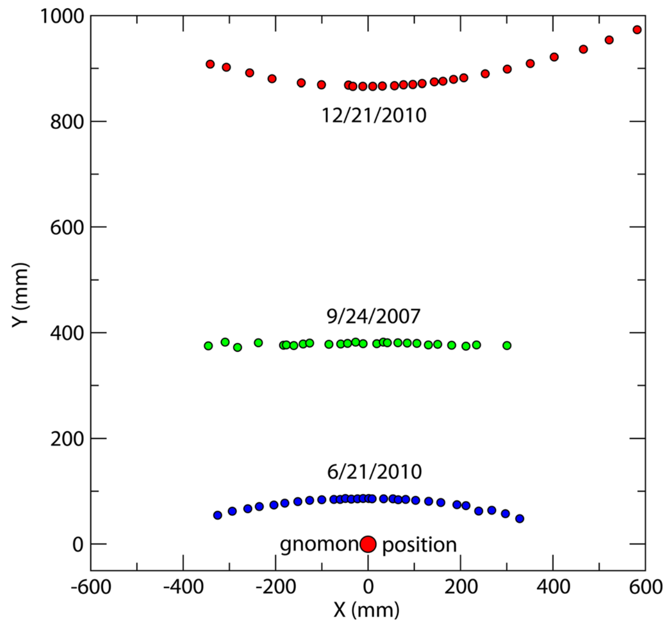


Fig. 3. The $X$-$Y$ positions of the end of the shadow of a 632 mm gnomon used at College Station, Texas. For the December 21, 2010 observations we scaled the coordinates by 632/550, because a 550 mm gnomon was used on that date. The curvature of these loci changes with the declination of the Sun over the course of the year. The shadow lengths and the $X$-positions, along with the hour angle of the Sun obtained from the times of the observations and the time of minimum shadow length, make it possible to derive the declination of the Sun. In this case it is best to use observations obtained when the Sun is roughly 1 h or more from the meridian.

Table I. Gnomon data. CDT and CST represent Central Daylight Time and Central Standard Time. The height of the Gnomon was 632 mm on June 21, 2010 and 550 mm on December 21, 2010. All lengths are in millimeters.

| CDT June 21 | $L$ | CST December 21 | $L$ |
|---|---|---|---|
| 11:25:00 | 330 | 11:07:45 | 844 |
| 11:35:00 | 300.5 | 11:15:00 | 829 |
| 11:46:30 | 268.5 | 11:25:00 | 807 |
| 11:55:00 | 245.5 | 11:35:15 | 787 |
| 12:06:17 | 217 | 11:49:20 | 770 |
| 12:15:04 | 196 | 11:59:20 | 761.5 |
| 12:25:02 | 172 | 12:10:00 | 756.5 |
| 12:35:01 | 151 | 12:15:00 | 754.5 |
| 12:45:06 | 131 | 12:20:00 | 754 |
| 12:55:00 | 112.5 | 12:25:15 | 754 |
| 13:00:10 | 104 | 12:30:00 | 755 |
| 13:05:00 | 99 | 12:36:00 | 756.5 |
| 13:10:47 | 92.5 | 12:40:38 | 759.5 |
| 13:15:00 | 88.5 | 12:45:18 | 761.5 |
| 13:20:00 | 86.5 | 12:50:00 | 765.5 |
| 13:25:00 | 86 | 12:55:43 | 771 |
| 13:30:10 | 86 | 13:00:00 | 775 |
| 13:37:20 | 92 | 13:05:20 | 782 |
| 13:45:00 | 101 | 13:10:06 | 788.5 |
| 13:49:30 | 106 | 13:20:00 | 805 |
| 13:56:00 | 117 | 13:30:00 | 824.5 |
| 14:04:30 | 132 | 13:40:00 | 848 |
| 14:15:00 | 154 | 13:50:00 | 875 |
| 14:25:00 | 175.5 | 14:01:20 | 910 |
| 14:38:12 | 206 | 14:10:50 | 946 |
| 14:45:00 | 224 | 14:20:06 | 987 |
| 14:55:00 | 247.5 | | |
| 15:05:00 | 275 | | |
| 15:15:00 | 303 | | |
| 15:25:00 | 331.5 | | |

Plots of the data from Table I are shown in Fig. 4. We have converted the time values to minutes since local apparent noontime (when $h = h_{\max}$).[10] In Fig. 4(a), the data look like they could be fit by a hyperbola, so that is what we show. It turns out that there is no simple function which fits such a data set. (We would need the latitude as part of the formulation, and that is the quantity we are trying to derive.) The best estimate of the minimum value in the summer solstice plot is obtained from a fourth order polynomial fit to a subset of the data (within 41 min of the meridian transit of the Sun). For the June 21, 2010 observations we used a gnomon of height $g = 632 \pm 1$ mm. The minimum shadow length is $L_{\min} = 85.84$ mm with a root-mean-square scatter of $\pm 0.64$ mm. $h_{\max}$ is equal to the arctangent of $g/L_{\min}$, or $82^\circ 15'.9 \pm 3.5$.

For December 21, 2010 we used a gnomon of height $g = 550 \pm 1$ mm. From a fourth order polynomial fit to all 26 points we find $L_{\min} = 753.98 \pm 0.74$ mm. $h_{\max}$ is found to be $36^\circ 06'.6 \pm 3'.4$ [see Fig. 4(b)].

Equation (4) gives $\phi = 30^\circ 48.7 \pm 3.5$, which is $11'.5$ north of the known value of $30^\circ 37'.2$ found from Google Earth. (1 arc min of latitude equals one nautical mile or 1852 m.) Equation (5) gives $\epsilon = 23^\circ 04'.7 \pm 3'.5$, which is several standard deviations smaller than the known value of $23^\circ 26'.2$. These values show what kind of results can be obtained with careful observations.

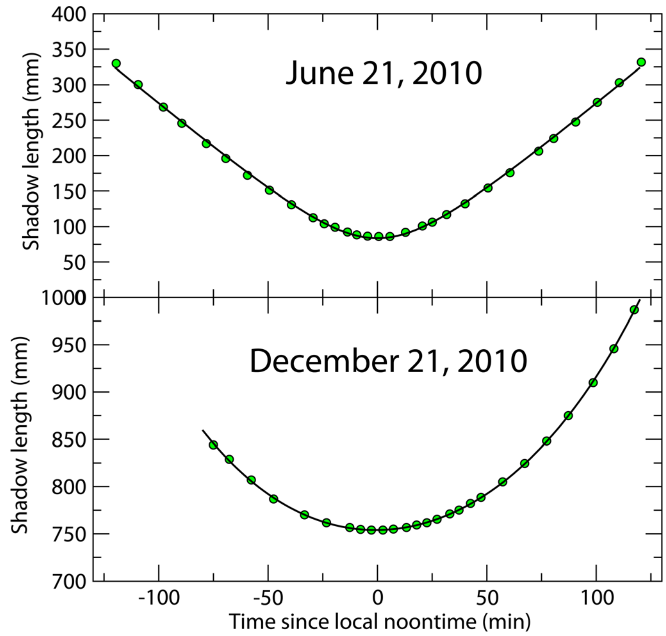


Fig. 4. (a) The shadow length of a 632 mm gnomon on the day of the summer solstice, as measured at College Station, Texas. For illustrative purposes a hyperbola is fit to the data. (b) The shadow length of a 550 mm gnomon on the day of the winter solstice, as measured at the same location. A fourth order polynomial is fit to the data.

So far we have used no information from the *Astronomical Almanac*.[7] To determine our longitude requires that we know the amount of time that the apparent Sun is ahead or behind the mean Sun. This difference is known as the *equation of time*.[11] It ranges from –14.2 min to + 16.4 min over the course of a year.

From four gnomon experiments done at College Station on November 12, 2006, September 24, 2007 and the two solstices mentioned we found that the mean Sun transits the meridian at 12:24:26 PM standard time, with an uncertainty of $\pm 72$ s. Because each degree of longitude corresponds to 4 min of time, our longitude is $96°06'.5 \pm 18'.1$. The result is within one standard deviation of the true value of $96°20'.4$.

## III. THE CIRCUMFERENCE OF THE EARTH

On September 3, 2006 we did a gnomon experiment in South Bend, Indiana, on the campus of the University of Notre Dame. For this experiment, we needed a value of the declination of the Sun from Ref. 7. Our location was found to be latitude $41°51'.5$, longitude $85°50'.5$. In late October of 2007, we drove from South Bend through Bella Vista, Arkansas, and on to College Station, keeping track of our route, mileage, and the length of any side trips. After subtracting the side trips, the elapsed distance on the odometer was 1248 statute miles. For a section of highway through central Illinois, we noted that 96.8 miles on the odometer corresponded to 98.0 miles according to the highway markers so our odometer exhibited a systematic error. On a subsequent occasion, we drove the final 92% of the exact same route with a different car, and the odometer mileage was 40 miles less (about 4%). We conclude that most odometers cannot be trusted. We adjusted the original mileage to $1248 \times (98.0/96.8) = 1263$ miles.

Next, we used a "map tool" and traced our route on a map in an atlas[12] and determined that if we could have traveled along a great circle route from South Bend to College Station, the length of the direct route was 0.744 of the length of the route we actually drove. Thus, the great circle distance from one place to the other was 940 statute miles.

Consider two locations on the Earth with (latitude, longitude) $= (\phi_1, \lambda_1)$ and $(\phi_2, \lambda_2)$, respectively. The length $\rho$ of the great circle arc between them can be obtained from the law of cosines

$$\cos(\rho) = \sin(\phi_1)\sin(\phi_2) + \cos(\phi_1)\cos(\phi_2) \cos(\lambda_2 - \lambda_1). \tag{6}$$

Given the latitudes and longitudes determined by the gnomon and clock at both locations, the great circle arc was $13.78°$. Thus, our estimate of the circumference of the Earth is $940 \times (360/13.78) = 24557$ miles, which is 1.4% smaller than the known value. Our value for the radius of the Earth is 6290 km compared with the known value of 6378 km. Given the simplicity of the tools we used, we achieved an accuracy far better than our expectations.

For comparison, we refer readers to a description of projects "Eratosthenes 2009" and "Eratosthenes 2010 America."[13] More than 15,000 students at more than 200 schools determined the radius of the Earth using this method and obtained 6290 km in 2009 and 6375 km in 2010.

## IV. THE DISTANCE TO THE MOON

### A. The method of Aristarchus

The method we used to determine the circumference of the Earth is in principle the same as that used by Eratosthenes. It was a generation before Eratosthenes that Aristarchus of Samos (ca. 310–230 BCE) determined the distance to the Moon in terms of the radius of the Earth.[14] Inherent in both methods is the idea that the Earth is, for all intents and purposes, spherical.

Aristarchus cleverly deduced that we can determine the distance to the Moon from the geometry of a lunar eclipse[14] (see Fig. 5). A lunar eclipse can occur only when the Moon is opposite the Sun. The shadow of the (spherical) Earth is circular in any plane perpendicular to the Earth-Sun line. Consider the triangle delineated by points A, C, and H in Fig. 5. AC is the radius of the Earth, $R_{\mathrm{earth}}$, and CH is the distance $d_{\mathrm{moon}}$ from the center of the Earth to the Moon. The horizontal parallax of the Moon, $P_M$, is given by

$$\sin P_M = \frac{R_{\mathrm{earth}}}{d_{\mathrm{moon}}}. \tag{7}$$

Thus, the distance to the Moon in Earth radii is $1/\sin P_M$. In Fig. 5, $P_S$ is the horizontal parallax of the Sun, $\sigma$ is the angular radius of the Sun, and $\tau$ is the angular radius of the Earth's shadow at the distance of the Moon. It is obvious that $\angle\mathrm{XCH} + P_S + P_M = 180°$. In other words, $180° - \angle\mathrm{XCH} = P_S + P_M$. Also, $\sigma + \angle\mathrm{XCH} + \tau = 180°$, which means that $180° - \angle\mathrm{XCH} = \sigma + \tau$. It follows that:

$$P_S + P_M = \sigma + \tau. \tag{8}$$

Aristarchus believed that the Sun was 19 times more distant than the Moon.[14] The true ratio is closer to 400.[15] Our value for the radius of the Earth and the distance to the Sun given in Sec. V means that the solar parallax $P_S \approx 0'.14$.

Aristarchus knew that the Moon and Sun had approximately the same angular diameter because the duration of a

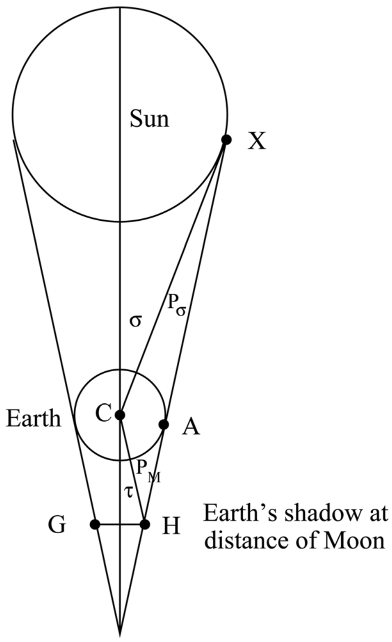


Fig. 5. Geometry of the lunar eclipse (not to scale). At point G the Moon is halfway into the shadow of the Earth. At H the Moon is halfway out of the shadow. Simple arguments and measurements originating with Aristarchus allow us to estimate the distance to the Moon in terms of the Earth's radius.

total solar eclipse is never more than a few minutes. He took the angular diameter of the Moon to be 0.5°. In the *Almagest*, Ptolemy states that the Earth's shadow at the distance of the Moon is 2.6 times the angular diameter of the Moon.[16] We used two measurements of the Sun made on May 1, 2010 and June 6, 2011 to obtain the angular diameter of $30'.8 \pm 0.9$, so $\sigma \approx 15'.4$.

We determined the Moon's angular diameter from sightings of the Moon through a 6.2 mm hole in a piece of cardboard which slides up and down a yardstick.[17] Our mean value for the angular diameter of the Moon was $31'.18$. The data in Ref. 17 and subsequent data, covering nearly 35 orbital periods, yield the time of the lunar perigee of 2011 May 14.70 UT, with a perigee-to-perigee period (the *anomalistic month*) of $27.5245 \pm 0.0467$ d, which is within one standard deviation of the known value of 27.55455 d. Based on our non-telescopic data, the lunar eclipse of June 15, 2011 occurred about 4.6 d after perigee, so the Moon's angular size would have been a bit larger than the mean value.

By using six images obtained from Ref. 18, we determined that the Earth's shadow was $2.56 \pm 0.03$ times the angular diameter of the full Moon during the lunar eclipse of June 15, 2011. Thus, $\tau = 0.5 \times 31.18 \times 2.56 \approx 39'.19$. It follows that $P_M = 15'.4 + 39'.19 - 0'.14 \approx 55'.17$. Equation (7) gives a lunar distance of $62.3 R_{\mathrm{earth}}$ with an error of at least $\pm 0.7 R_{\mathrm{earth}}$. The known mean distance is $60.27 R_{\mathrm{earth}}$.

### B. Using the motion of the Moon against the stars

We have seen that the geometry of a lunar eclipse allows a determination of the distance to the Moon. Hipparchus (ca. 140 BCE) used observations of the solar eclipse of 14 March 190 BCE to derive the distance to the Moon to be between 71 and $83 R_{\mathrm{earth}}$.[19] Can we determine the Moon's distance without the use of an eclipse? Simultaneous observations of the Moon's position from two widely separated places on the Earth would suffice, as can be done from two observatories widely separated in latitude but on the same meridian of longitude.[20]

Observations of the Moon from the same location over a night can also be used to determine the distance to the Moon. In this case, we use a large fraction of the Earth's diameter as a baseline. The complication is that while we wait for the Earth to rotate, the Moon is also orbiting the Earth.

The Moon's orbital period with respect to the background stars is 27.32 d. Thus, on average, the Moon moves 0.55° east per hour against the background of stars as viewed by an observer at the Earth's center. Because of the parallax of the Moon, the observed motion of the Moon against the stars when it is high in the sky, as observed by someone on the surface of the Earth, is roughly $(1/6)^\circ$ per hour less.[21]

We need the geocentric versus topocentric shifts in right ascension and declination, respectively, of the Moon due to our location on the surface of the Earth. The *topocentric* coordinate system has the observer on the Earth's surface at the origin. Let $\Delta\alpha = \alpha' - \alpha = t - t'$. (Right ascension increases to the east, and hour angle increases to the west.) Then, Eq. (35) of Chap. 9 of Ref. 20 gives

$$\tan \Delta\alpha = -\frac{R_{\mathrm{earth}}}{d_{\mathrm{moon}}} \frac{\sin t \cos \phi'}{\cos \delta - \left(\dfrac{R_{\mathrm{earth}}}{d_{\mathrm{moon}}}\right) \cos t \cos \phi'}. \tag{9}$$

If we let $\delta' = \delta + \Delta\delta$ and $T = \tan \Delta\delta$, Eq. (38) of Chap. 9 of Ref. 20 becomes

$$\frac{\tan \delta + T}{1 - T \tan \delta} = \frac{\cos t' \left[\sin \delta - \left(\dfrac{R_{\mathrm{earth}}}{d_{\mathrm{moon}}}\right) \sin \phi'\right]}{\cos \delta \cos t - \left(\dfrac{R_{\mathrm{earth}}}{d_{\mathrm{moon}}}\right) \cos \phi'}. \tag{10}$$

The primed quantities are the topocentric values of the declination, hour angle, and latitude, and the unprimed values are for a hypothetical observer at the Earth's center. Equations (9) and (10) cannot be easily inverted to give $d_{\mathrm{moon}}/R_{\mathrm{earth}}$. But because these equations contain this ratio, we can show how single-site observations can be used to obtain the distance to the Moon. What follows is primarily a demonstration of the method.

In Fig. 6, we show the rate at which the Moon's apparent position (for an observer situated at College Station) varies as a function of hour angle for several occasions over nearly the full range of distance of the Moon from the Earth. Not surprisingly, the Moon moves against the background of stars more slowly on a night when it is near apogee than when it is near perigee.

In Fig. 7, we show a simple cross staff.[22] By using such a device we can obtain the angular separation of two objects in the sky. For two objects with known right ascensions and declinations $\rho$ is given by an expression similar to Eq. (6)

$$\cos \rho = \sin \delta_1 \sin \delta_2 + \cos \delta_1 \cos \delta_2 \cos(\alpha_2 - \alpha_1). \tag{11}$$

If the Moon's angular separation from two stars of known right ascension and declination is determined, it is possible to determine the right ascension and declination of the Moon. The easiest way to envision this determination is to take a star chart and a compass and draw two circular arcs of different radii centered at the locations of two particular stars. We obtain two numerical solutions of the Moon's

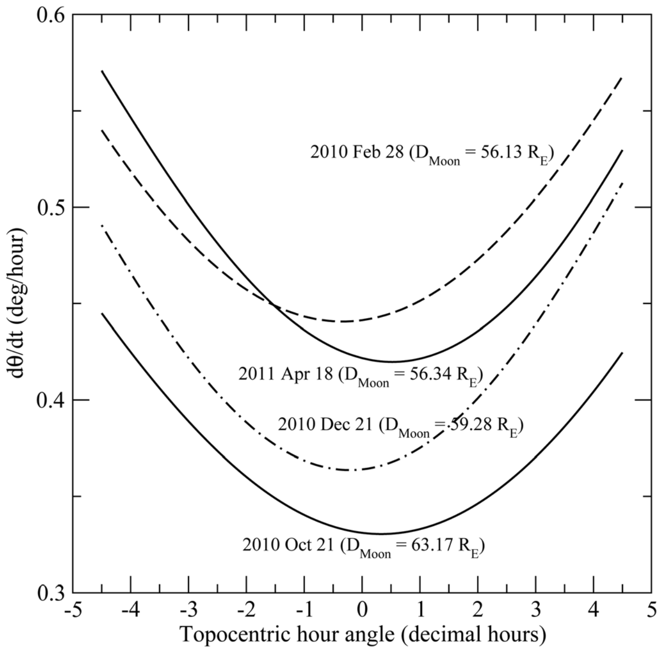


Fig. 6. The angular motion of the Moon as a function of hour angle for an observer at College Station.

Table II. Angular separations (May 15, 2011).

| UT | Object pair | $\rho_{obs}$ | $\rho_{true}$ | $\rho_{true} - \rho_{obs}$ |
|---|---|---|---|---|
| 02:21:00 | $\alpha$ Vir versus Moon | 3.°41 | 3.°70 | +0.° 29 |
| 02:24:00 | Saturn versus Moon | 12.85 | 13.21 | +0.36 |
| 02:28:00 | $\delta$ Crv versus Moon | 10.71 | 10.85 | +0.14 |
| 02:30:00 | Saturn versus $\alpha$ Vir | 13.60 | 13.71 | +0.11 |
| 04:04:00 | $\alpha$ Vir versus Moon | 3.28 | 3.40 | +0.12 |
| 04:06:40 | Saturn versus Moon | 13.34 | 13.86 | +0.52 |
| 04:08:00 | $\delta$ Crv versus Moon | 11.33 | 11.34 | +0.01 |
| 04:10:00 | Saturn versus $\alpha$ Vir | 13.63 | 13.71 | +0.08 |
| 05:54:00 | $\alpha$ Vir versus Moon | 3.32 | 3.22 | –0.10 |
| 05:56:00 | Saturn versus Moon | 14.52 | 14.53 | +0.01 |
| 05:58:30 | $\delta$ Crv versus Moon | 11.77 | 11.91 | +0.14 |
| 06:01:40 | Saturn versus $\alpha$ Vir | 13.55 | 13.71 | +0.16 |
| 06:04:00 | $\gamma$ Hya versus Moon | 9.33 | 9.31 | –0.02 |
| 08:02:00 | $\alpha$ Vir versus Moon | 3.63 | 3.15 | –0.48 |
| 08:03:00 | Saturn versus Moon | 14.80 | 15.34 | +0.54 |
| 08:05:00 | Saturn versus $\alpha$ Vir | 13.55 | 13.71 | +0.16 |
| 08:31:00 | Saturn versus $\alpha$ Vir | 13.60 | 13.71 | +0.11 |
| 08:38:00 | $\alpha$ Vir versus Moon | 3.35 | 3.17 | –0.18 |
| 08:41:00 | Saturn versus Moon | 15.06 | 15.60 | +0.54 |

position, one on each side of the great circle arc joining the two stars. One of those possibilities can easily be eliminated. Once an approximate location of the Moon's position is found, we can use a computer program to search a box that covers a range of right ascension and declination to find the celestial location which is the number of degrees from each of the two stars of known position.

Table II gives five sets of observations made on the night of May 15, 2011 UT, an occasion when the Moon was close to the bright star $\alpha$ Vir and within a day of perigee (Fig. 8). On this night, the Moon was waxing and 93% illuminated and was visible almost the entire night. One of our reference "stars" was Saturn. Given the accuracy of our observations we can assume that the position of Saturn was constant on this night. As we can see, the angular separation of Saturn and $\alpha$ Vir was measured to better than 0.2° on five occasions. Because the Moon is not a point source, measuring the angular separation of the Moon and Saturn or the Moon and a star is more difficult than measuring the angular separation of two bright stars or one bright star and a planet.

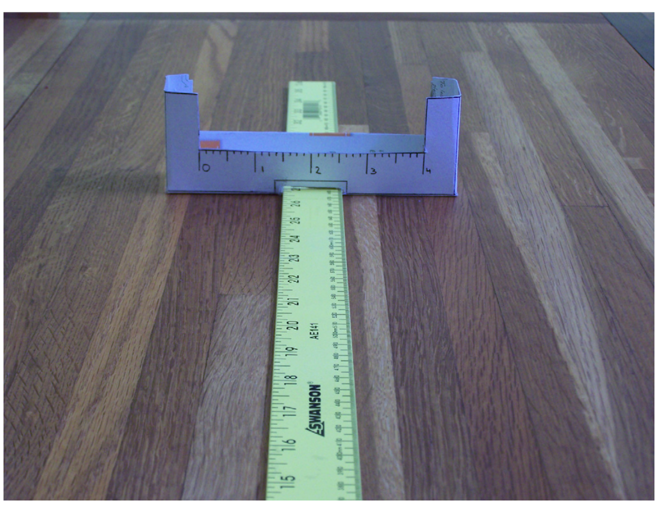


Fig. 7. The cross staff. The cardboard cross piece slides up and down the yardstick. By using simple geometry we can use this device to determine the angular separation of objects in the sky.

In Table III, we give the derived right ascensions and declinations of the Moon from our observations on May 15, 2011. At the end of the night, the reference stars $\delta$ Crv and $\gamma$ Hya were too low in the sky to be seen. The uncertainties of the right ascension and declination of the Moon at 08:03 and 08:41 UT were derived assuming an uncertainty of $\pm 0.3°$ for the angular separation of Saturn and the Moon and $\alpha$ Vir and

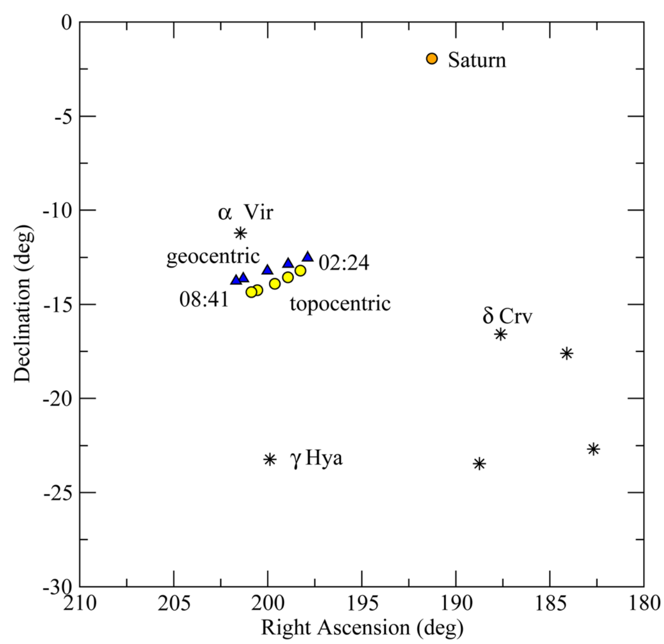


Fig. 8. The topocentric and geocentric right ascension and declination of the Moon on May 15, 2011. The topocentric values are calculated for College Station. The zenith angle of the Moon is greater for an observer situated on the surface of the Earth compared to a hypothetical observer at the center of the Earth. In other words, by observing on the surface of the Earth, the Moon appears to be lower in the sky compared to what would be seen by an observer at the center of the Earth. This elevation angle offset converts to varying shifts in right ascension and declination over the course of the night.

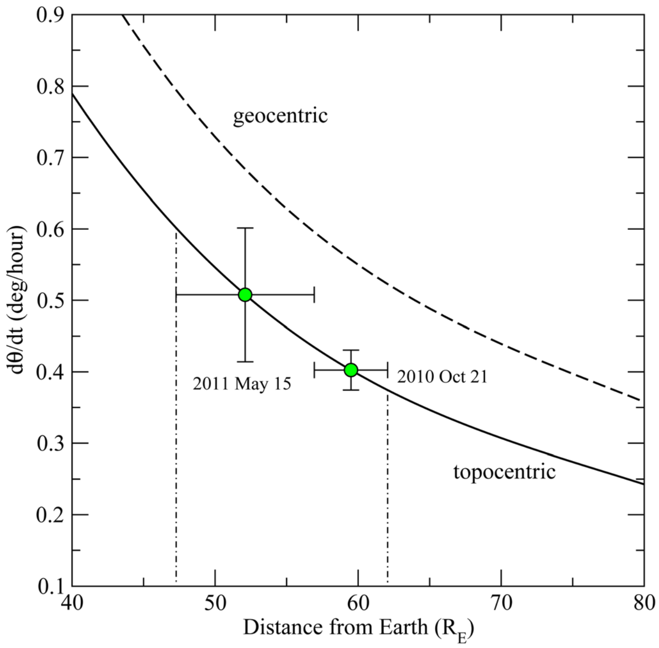


Fig. 9. The angular motion (dashed curve) of an Earth-orbiting object as viewed by a hypothetical observer at the Earth's center, as a function of its distance in $R_{\rm earth}$. The mean angular motion (solid curve) of an Earth-orbiting object as viewed from College Station. The average is taken over 8 h centered on the meridian transit. On October 21, 2010 we measured the Moon to move $0.40\pm0.03^{\circ}$/h from six observations over 9.1 h. The implied value of the Moon's distance is between 57 and $62R_{\rm earth}$ on that occasion. On May 15, 2011 we measured the Moon to move $0.508\pm0.094^{\circ}$/h from three sets of observations taken over 3.9 h. The implied distance to the Moon is roughly 47 to $58R_{\rm earth}$.

the Moon. Because of the large uncertainty in the right ascension and declination of these final two observations, we exclude data sets four and five from further analysis. The first, second, and third determinations of the position of the Moon yield an apparent angular rate of $(0.508\pm0.094)^{\circ}$/h. The corresponding range of the Moon's distance is 47.3 to $58.5R_{\rm earth}$ with a most likely value of $52.1R_{\rm earth}$.

October 21, 2010 UT was the day after the lunar apogee. On this day, we made a series of six measurements of the angular separation of the Moon and Jupiter over 9.1 h. (No stars near the Moon were bright enough to be seen with the unaided eye given the quality of the sky.) Thus, we could not use the same method to obtain multiple estimates of the right ascension and declination of the Moon at a given time. The separation of the Moon and Jupiter increased from 11.31 to 14.77°, with a mean rate of increase of $(0.403\pm0.028)^{\circ}$/h. If we assume that the Moon was moving directly away from the position of Jupiter (which was only approximately true), this angular rate is a lower limit to the rate of change of position of the Moon against the background of stars. The implied range of lunar distance is 57.2 to $62.1R_{\rm earth}$, with a most likely value of $59.5R_{\rm earth}$.

If we take the one standard deviation upper limit of the rate of change of the position of the Moon on May 15, 2011 and the one standard deviation lower limit of the rate of change of the lunar position on October 21, 2010, we find a conservative estimate of the range of the Moon's distance, namely, 47 to $62R_{\rm earth}$ (Fig. 9). The observations and analysis required for this method of estimating the Moon's distance are much more complicated than Aristarchus's method using the geometry of a lunar eclipse.

Table III. Derived and true topocentric positions of Moon (May 15, 2011). Also, the angular distance between the derived position of the Moon based on observations with the cross staff versus the true topocentric position of the Moon. For the fourth and fifth determinations the uncertainties in the right ascension and declination derive from the assumption that the angular separations of the Moon versus α Vir and the Moon versus Saturn were accurate to ±0.3°.

| UT | $\alpha_{\rm obs}$ | $\delta_{\rm obs}$ | $\alpha_{\rm true}$ | $\delta_{\rm true}$ | $\rho$ |
|---|---|---|---|---|---|
| 02:24:00 | 198.°15±0.09 | −12.°76±0.09 | 198.°27 | −13.°20 | 0.°46 |
| 04:04:40 | 198.76±0.04 | −13.14±0.07 | 198.93 | −13.56 | 0.45 |
| 05:56:00 | 199.53±0.02 | −13.94±0.01 | 199.63 | −13.90 | 0.10 |
| 08:03:00 | 199.50±0.57 | −14.31±0.36 | 200.55 | −14.25 | 1.02 |
| 08:41:00 | 200.03±0.62 | −14.27±0.34 | 200.87 | −14.34 | 0.81 |

## V. THE DISTANCE TO THE SUN

In 2008, we were participants in the Summer Science Program, a residential non-credit enrichment program for incoming high school juniors and seniors. Coauthors Krisciunas and Kim were faculty, and the other coauthors were students. The tradition of the Summer Science Program is to divide the students into teams of three for the observing. Each student writes computer code to determine the orbital parameters of a particular asteroid. In Table IV, we give the orbital solution of asteroid 8567 (= 1996 HW1) by coauthor Steeger, along with the parameters obtained from Ref. 23. The code to determine the orbital parameters was debugged using observations of Ceres and its orbital parameters given by Ref. 23. Note that our orbit solution is based on observations extending over a very small arc of the full orbit. (The orbital period of asteroid 1996 HW1 is 2.93 y.)

A goal of the Summer Science Program has been to observe a near-Earth object simultaneously from two sites so that its distance could be derived, the size of the Astronomical Unit could be measured, and the scale of the solar system be determined. We report one such experiment here.

On July 24, 2008 UT we took images of asteroid 1996 HW1 at the Etscorn Observatory of the New Mexico Institute of Mining and Technology in Socorro, New Mexico, using a 15 cm Takahashi refractor and a CCD camera. The geographic position was latitude $+34^{\circ}04'21''.7$, longitude W $106^{\circ}54'50''.1$. A simultaneous image was obtained in Ojai, California, at latitude $+34^{\circ}26'04.0$, longitude W $119^{\circ}11'22''.6$ using a 25 cm Meade reflector and CCD camera. Figures 10 and 11 show the asteroid at 08:17:28 UT on July 24, 2008 as viewed from the two sites.

Table IV. Steeger orbit determination for asteroid 1996 HW1. The mean anomaly increases by 0.33676929° per day according to JPL Horizons, so it is 334.°2791 at the epoch of the Steeger solution.

| Parameter | Description | Steeger value | JPL Horizons value |
|---|---|---|---|
| $t_0$ | Epoch (UT) | 2008 June 29.35 | 2007 January 15.00 |
| $t_0$ | Epoch (Julian Date) | 2454646.85 | 2454115.5 |
| $M$ | Mean anomaly at $t_0$ | 335.°1461 | 155.°33678 |
| $a$ | Semi-major axis | 2.0855 AU | 2.046041 AU |
| $e$ | Eccentricity | 0.4575 | 0.449165 |
| $i$ | Inclination angle | 8.°5033 | 8.°437363 |
| Ω | Longitude of ascending node | 177.°6887 | 177.°216737 |
| ω | Argument of perihelion | 176.°1618 | 177.°020070 |

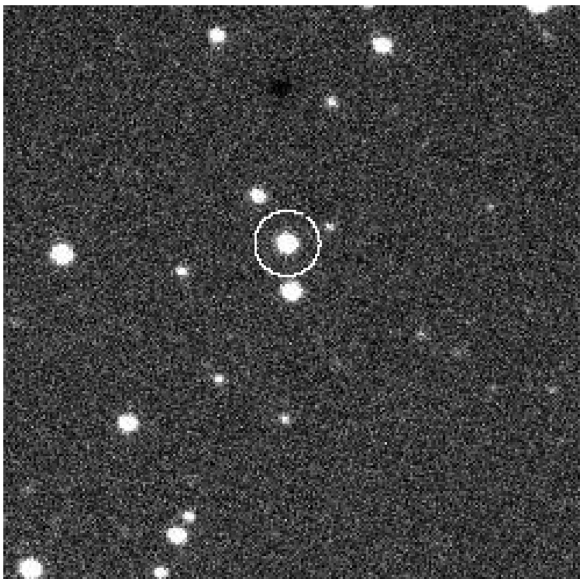

Fig. 10. Asteroid 1996 HW1 is circled in this unfiltered 90 s image obtained by DeBenedictis with a 15 cm Takahashi refractor at the Etscorn Observatory, Socorro, New Mexico, on July 24, 2008 at 08:17:27.8 UT. North is up, east is to the left.

The right ascensions and declinations of a number of field stars were determined using the astronomical imaging and data visualization program DS9 (made available by the Smithsonian Astrophysical Observatory) and an image of the field obtained from the Space Telescope Science Institute Digital Sky Survey.[24] We then determined a transformation from pixel coordinates to right ascension and declination using the IRAF programs CCMAP and CCTRANS[25] and determined the right ascension and declination of the asteroid for each of our images. From the New Mexico site we determined the asteroid's topocentric position to be $\alpha = 21:27:07.42$ and $\delta = +15^\circ 53' 02''.77$. From the Etscorn image and the positions of the field stars, the root-mean-square errors in the right ascension and declination were $\pm 0.31''$ and $\pm 0.36''$, respectively. Our image obtained in California just barely has the asteroid in the frame, and the guiding was not as good. Still, we found a topocentric position of $\alpha = 21:27:07.77$ and $\delta = +15^\circ 53' 02''.25$. The root-mean-square errors of the field star positions were $\pm 0.52''$ and $\pm 0.44''$, respectively, for right ascension and declination.

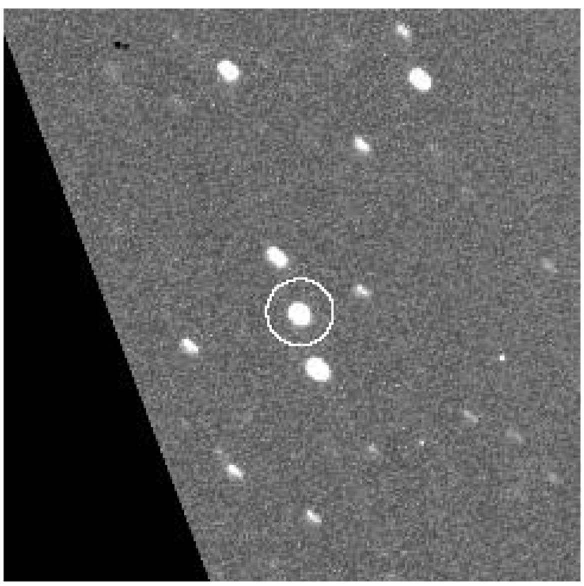

Fig. 11. Asteroid 1996 HW1 is circled in this unfiltered 90 s image obtained by Tabak and Pasricha with a 25 cm Meade reflector at Besant Hill School in Ojai, California, on July 24, 2008 at 08:17:28 UT, with an uncertainty of no more than $\pm 2$ s. North is up, east to the left. This image was taken at the same time as Fig. 10, but the asteroid is roughly 5 arc sec to the left (east) as observed at the other end of a baseline of 1130 km.

The effect on right ascension and declination due to parallax and the finite size of the Earth can be calculated much more easily than using Eqs. (9) and (10) if the planet or asteroid under consideration has a distance $d$ considerably greater than the distance to the Moon. Smart gives the relevant equations[26]

$$\Delta\alpha = -\frac{R_{\rm earth}}{d}\sin t\cos\phi'\sec\delta, \tag{12}$$

$$\Delta\delta = -\frac{R_{\rm earth}}{d}(\sin\phi'\cos\delta - \cos\phi'\sin\delta\cos t). \tag{13}$$

The values on the left-hand sides of Eqs. (12) and (13) are in radians.

Given that the latitudes of the New Mexico and California sites are almost the same, we have almost no leverage to use Eq. (13) to determine the distance to the asteroid. We limit ourselves to a consideration of the effect of parallax on the right ascensions. Let the New Mexico site be "position 1" and the California site be "position 2." Consider the seconds part of the observed right ascension of the asteroid. There exists corrections to the right ascension such that $7.77 - c_2 = 7.42 - c_1$. The corrections $c_i$ adjust the observed right ascensions to what would be observed by a hypothetical observer at the center of the Earth. So, 0.35 s equals the difference of the parallactic corrections. Because one second of time in right ascension equals $15\cos\delta$ arc sec, we have $5''.05 \pm 0.61 = c_2 - c_1$. The uncertainty comes from the square root of the sum of squares of the uncertainties of the pixel to right ascension/declination transformations from CCMAP in IRAF.

At the New Mexico site at the time Fig. 10 was taken the hour angle of the asteroid was $-1.899^\circ$, and at the California site the hour angle of the asteroid was $-14.176^\circ$ when Fig. 11 was taken. By using Eq. (12), we obtain a distance to the asteroid of

$$d = \frac{R_{\rm earth}(0.210010 - 0.028541)}{\left(\frac{5.05 \pm 0.061}{206265}\right)}. \tag{14}$$

The number of arc seconds in a radian is 206265. If we adopt our value of the radius of the Earth from Sec. III (namely, 6290 km), we obtain a distance to the asteroid of $(4.66 \pm 0.56) \times 10^7$ km.

Using the Steeger orbit solution in Table IV, the method of Meeus,[27] and the rectangular coordinates of the Sun on July 24, 2008 from the *Astronomical Almanac* for 2008 (Ref. 7) we determined that asteroid 1996 HW1 was 0.294 AU distant when Figs. 10 and 11 were taken. Our resulting value for the Astronomical Unit is $(1.59 \pm 0.19) \times 10^8$ km, which is roughly 6% larger than the accepted value of $1.496 \times 10^8$ km.

Our value of the solar parallax, $P_S$ in Fig. 5, equals $(6290/1.59 \times 10^8) \times 206265 = 8''.2$, or about $0'.14$, which we used in Sec. IV A. Our distance to the Sun in Earth radii divided by our distance to the Moon in Earth radii gives a Sun distance that is roughly 406 times the distance to the Moon, which is close to the known mean value of 389.

## VI. SYSTEMATIC AND RANDOM ERRORS

Very often a set of measurements exhibits a Gaussian distribution. This distribution allows us to identify outliers in the data. If some data value is several standard deviations from the mean (or expected) value, we have either underestimated the size of our random errors or there is some unaccounted source of systematic error.

In Sec. II, we derived a value of the latitude for a particular location in College Station, which was $11'.5$ north of the known value. Usually we do not know the "true" value, but in this case we do. From a scatter of the measurements of the shadow length of our gnomon we could obtain a numerical value of the root-mean-square scatter of the data about some best fit line. That gives us an estimate of the uncertainty of the minimum shadow length, which translates into a random error for the maximum elevation angle of the Sun on some given day, which then leads to a random error for our value of the latitude. Our value of the latitude is within three standard deviations of the known value. Our value of the tilt of the Earth's axis of rotation to the plane of its orbit is six standard deviations from the true value. Thus, we have room for improvement in the experiment.

Some sources of uncertainty have not yet been mentioned. For the location of the June 21 and December 21 gnomon experiments we measured the levelness of the spot where the data were taken, finding that a perfectly straight gnomon which was perfectly squarely set in its base would have been tilted $6\pm1$ arc min north of the zenith. This would have made the elevation angle too low by that amount at local noontime, but less so at other times. To complicate matters, our gnomon is a wooden dowel rod which is not totally straight, and we do not know just how squarely it sits in the block of wood that is its base.

Regarding the use of the map tool in Sec. III, we found that the great circle arc from South Bend to College Station, was $0.7437\pm0.0009$ of the route driven. We rounded this value to 0.744 and decided to ignore the numerical uncertainty.

In Sec. IV A we found from six images of the Moon in partial lunar eclipse taken from 21:29:55 to 21:52:59 UT that the angular diameter of the Earth's shadow was 2.46, 2.65, 2.49, 2.62, 2.56, and 2.56 times the angular diameter of the Moon. These values were obtained using hard copies of the images, a compass, and a ruler. In all instances we could see more than a 180 degree arc of the illuminated part of the Moon. The mean ratio was $2.56\pm0.03$. To find angle $\tau$, the angular radius of the Earth's shadow, we used the more robust mean value of the angular diameter of the Moon,[17] rather than an estimated value of the Moon's angular size on June 15, 2011.

In Sec. IV B and Table II we give the results of an experiment that is at or beyond the capabilities of a hand-held ruler plus cardboard cross piece. We found that the systematic error of the angular separation of two bright point sources (Saturn versus $\alpha$ Vir) was $+0.12^\circ$. The random error was $\pm0.04^\circ$. This result is considerably better than the typical uncertainty of $0.20^\circ$ to $0.25^\circ$ for non-telescopic measurements.[28] But consider our measurements of the angular separation of Saturn and $\alpha$ Vir compared to the Moon. Our mean systematic error is as great as $0.39^\circ$ and our random error is $\pm0.23$ to $\pm0.29^\circ$. Table III shows that if we measure the Moon's position with respect to three or more point sources, we can locate the Moon's location on the sky to within $0.3^\circ$ on average. If we measure the Moon's position with respect to only two point sources, the error in position is too large to be useful for determining the rate of change of the Moon's position over a night.

There are other errors that are neither systematic nor random. In Fig. 4(a) we fitted a hyperbola to the full data set. This fit is for illustrative purposes only. Careful scrutiny shows that the points do not randomly scatter above and below the curved line. The actual function we need requires the latitude of the site, which is the quantity we are trying to derive. In this case we fitted a fourth order polynomial to a subset of the data around the time of the minimum shadow length. The point is that we fit some data under the *assumption* that we have the appropriate function. The most appropriate function also depends on the typical size of the error bars of the data points. If the error bars are large, high order polynomial fits would not be justified.

## VII. DISCUSSION AND CONCLUSIONS

By using a vertical stick, a car, watch, map, and a map tool, we measured the size of the Earth from first principles. We used observations of the elevation angle of the Sun over a duration of a few hours overlapping local noontime on the summer solstice and the winter solstice of 2010 to determine the latitude and longitude of College Station. We also determined the latitude and longitude of South Bend and obtained the radius of the Earth to be 6290 km, about 1.4% too small.

Aristarchus's method of determining the distance to the Moon derives from the geometry of the Earth, Moon, and Sun at the time of a lunar eclipse. One key observable is the angular size of the Earth's shadow at the distance of the Moon compared to the angular size of the Moon. By using available images[18] of the lunar eclipse of June 15, 2011 we found that the Earth's shadow was 2.56 times the angular diameter of the Moon. The corresponding distance of the Moon is 62.3 Earth radii, about 3.3% larger than the known mean value.

It is also possible to determine the distance to the Moon from observations of its motion against the background of stars. These observations are made from a single site over one night. This method is much more complicated than Aristarchus's method, but these observations showed that the Moon was between 47 and $62R_{\rm earth}$ distant. The known range of the Moon's distance is 55.9 to $63.8R_{\rm earth}$.

Simultaneous observations from Socorro, New Mexico, and Ojai, California, allowed us to determine the distance to the asteroid 1996 HW1 on July 24, 2008. A determination of the orbital elements of this asteroid by one of us allowed us to calculate that the asteroid was 0.294 AU distant on that date. The final result was a calibration of the Astronomical Unit 6% larger than its known value.

Given the seeing and tracking constraints associated with our asteroid observations, our asteroid experiment was at the limits of our small telescopes. To determine the distance to a main belt asteroid (which would have been ten times more distant) would have required a baseline ten times bigger than the 1130 km baseline we used if we wanted the parallax to be several arc seconds. Such an experiment would be nearly impossible on the surface of the Earth.

Our qualitative results are easy to understand and are worth restating. Using inexpensive equipment we determined the size of the Earth and the distance to the Moon. To determine the Astronomical Unit we did not need to organize an international endeavor such as was done for the Venus transits of 1761, 1769, 1874, and 1882.[4] We can measure the length of the Astronomical Unit with carefully timed observations of a near-Earth asteroid using telescopes comparable with those owned by many amateur astronomers.

It is also worth recalling why astronomers have such an obsession with determining cosmic distances. Thanks to Kepler's third law the length of the Astronomical Unit yields the distances to all other objects that orbit the Sun. Once we have the distances to stars like the Sun via trigonometric stellar parallaxes—the next rung of the distance ladder—we can use photometric methods to determine distances to star clusters using the fact that main sequence stars of the same mass have comparable intrinsic brightness. A simple equation relates the apparent brightness of a star to its intrinsic brightness and the distance. We can exploit the Cepheid period-luminosity relation to calibrate our way across the Galaxy and to nearby galaxies. Knowing the energy budget of a star helps us to determine what it is made of, its structure, how long it will live, and how it will die. Luminosities of Type Ia supernovae have allowed astronomers to address some the largest questions we can ask, such as the ultimate fate of the universe.

The interconnectedness of astronomical topics means that the big questions are related to the Earth size experiment of Eratosthenes carried out more than 22 centuries ago.

## ACKNOWLEDGMENTS

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[1]The "van of Eratosthenes," refers to the Greek astronomer Eratosthenes (ca. 276–195 BCE) who obtained the first estimate of the Earth's circumference from observations of the Sun's elevation on the summer solstice from Alexandria and a town 7° to the south.

[2]M. Rowan-Robinson, *The Cosmological Distance Ladder: Distance and Time in the Universe* (W. H.Freeman, New York, 1985).

[3]O. Gingerich and J. R. Voelkel, "Tycho Brahe's Copernican campaign," J. Hist. Astron. **29**, 1–34 (1998).

[4]A. Van Helden, "Measuring solar parallax: The Venus transits of 1761 and 1769 and their nineteenth-century sequels," in *Planetary Astronomy from the Renaissance to the Rise of Astrophysics, Part B: The Eighteenth and Nineteenth centuries*, edited by R. Taton and C. Wilson (Cambridge U.P., Cambridge), 1995, pp. 153–168.

[5]Reference 2, p.48.

[6]A. G. Riess et al., "A 3% solution: Determination of the Hubble constant with the Hubble Space Telescope and Wide Field Camera 3," Astrophys. J. **730**, 119-1–18 (2011).

[7]*The Astronomical Almanac* (Nautical Almanac Office, Washington, DC, 2006). For this article we also needed information from the 2008 and 2010 volumes.

[8]The celestial meridian is the imaginary line in the sky that separates the east half from the west half.For an observer in the northern hemisphere the meridian extends from the north point on the horizon through the North Celestial Pole, through the zenith and down to the south point on the horizon. An object such as the Sun is at its highest point above the horizon when it crosses the celestial meridian.At that moment the local apparent solar time is, by definition, exactly 12 noon.

[9]If a vertical pointed stick is used, one obtains the elevation angle of the upper limb of the Sun. To find the elevation angle of the center of the Sun requires subtracting the angular radius of the Sun.

[10]Apparent solar time is related to the hour angle of the visible Sun in the sky.Due to the tilt of the Earth's axis of rotation to the plane of its orbit and the Earth's elliptical orbit, apparent solar time ranges from 14 min behind to 16 min ahead of mean solar time. Our watch time is mean solar time adjusted (by the longitude difference) to the nearest 15° line of longitude west of Greenwich, England. For example, Central Standard Time is for locations which are about 90° longitude west of Greenwich, or 6 h west.

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[12]*The New International Atlas* (Rand McNally, Chicago, 1980).

[13]V. Bekeris et al., "Eratosthenes 2009/2010: An old experiment in modern times," Astron. Educ. Rev. **10**, 010201 (2011).

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[16]G. J. Toomer, *Ptolemy's Almagest* (Springer-Verlag, Berlin, 1984), p. 254.

[17]K. Krisciunas, "Determining the eccentricity of the Moon's orbit without a telescope," Am. J. Phys. **78**, 834–838 (2010).

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[20]W. M. Smart, *Textbook on Spherical Astronomy*, 6th ed., revised by R. M. Green (Cambridge U.P., Cambridge, 1977), pp. 200–202.

[21]Reference 14, pp. 252–254.

[22]The pattern for the device that slides up and down the yardstick was obtained from the University of Washington, <www.astro.washington.edu/courses/labs/clearinghouse/labs/Skywatch/angles.html>.

[23]The JPL Horizons website is at <ssd.jpl.nasa.gov/horizons.cgi#top>.

[24]The website of the Digital Sky Survey is at <archive.stsci.edu/cgi-bin/dss_form>.

[25]The software IRAF is distributed by the National Optical Astronomy Observatory, which is operated by the Association of Universities for Research in Astronomy.

[26]Reference 20, pp. 209–210.

[27]J. Meeus, *Astronomical Formulae for Calculators*, 4th ed. (Willmann-Bell, Richmond, VA, 1988), pp. 125–130.

[28]K. Krisciunas, "A more complete analysis of the errors in Ulugh Beg's star catalogue," J. Hist. Astron. **24**, 269–280 (1993). Only Tycho Brahe (1546-1601), his engineers, and observers were able to design and use metal instruments that gave much more accurate results. Nine fundamental reference stars observed by Tycho and his observers are accurate to better than 1 arc min in right ascension and declination. See J. L. E. Dreyer, *Tycho Brahe: A Picture of Scientific Life and Work in the Sixteenth Century* (Peter Smith, Gloucester, MA, 1977), pp. 387–388.